

March 1996
CERN-TH/96-74
PUPT-1604

BPS SPECTRUM OF THE FIVE-BRANE AND BLACK HOLE ENTROPY

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ABSTRACT

We propose a formulation of 11-dimensional M-theory in terms of five-branes with closed strings on their world-volume. We use this description to construct the complete spectrum of BPS states in compactifications to six and five dimensions. We compute the degeneracy for fixed charge and find it to be in accordance with U-duality (which in our formulation is manifest in six dimensions) and the statistical entropy formula of the corresponding black hole. We also briefly comment on the compactification to four dimensions.

INTRODUCTION.

One of the outstanding challenges that one faces in building a successful theory of quantum gravity is to provide a microscopic description of black holes that explains and reproduces the Hawking-Bekenstein formula for the entropy [1]. This question has been hard to answer within string theory because, until recently, black holes only arose as particular classical solitons of the low energy effective field theory and hence could only be studied in string perturbation theory. This situation has changed quite dramatically through the recent developments related to string duality [2, 3]. It has been discovered that black holes and other RR-solitons have an exact description in string theory in terms of D-branes [4], and that their quantum states are related by duality to elementary strings excitations. These facts have been used recently in [5] to give a microscopic description in terms of D-branes of certain five-dimensional extremal black holes and to check that it reproduces the expected entropy formula (see also [6], and for extensions to 4 dimension [7]). The central idea of this computation is a comparison of the asymptotic growth of the number of BPS states in string theory with a given charge and the area or volume of the horizon of the corresponding black hole geometry. Specifically, the statistical entropy $S(Z)$ as a function of the charge Z is expected to behave as

$$S(Z) = 2\pi|Z|^\alpha \tag{1}$$

with $\alpha = 2, \frac{3}{2}, 1$ for dimension $d = 4, 5, 6$ respectively. The precise form of the right-hand side is further restricted by the required invariance under the U-duality symmetry group.

U-duality and other string dualities are still rather mysterious in the D-brane description, because it necessarily makes a distinction between charges that correspond to RR-fields and those of the NS-sector of the theory. On the other hand, there are many indications that elementary strings and all other p -branes have a unified description in 11 dimensions [8]. One therefore expects that the U-duality invariance of the BPS spectrum can be explained by extending the D-brane analysis to this eleven-dimensional M-theory [3, 9, 10]. It is known that M-theory contains membranes and five-branes, which are charged relative to the three-form gauge field C_3 and its six-form dual \tilde{C}_6 respectively. However, by forming bound states the five-brane can also carry C_3 charges and it therefore seems the natural starting point for a unified treatment of all BPS states in string theory.

It has been suggested that the five-brane can be viewed as a D-object on which membranes can have boundaries [11]. In this picture the world-volume theory of the five-brane is induced by the boundary states of the membrane and thus is naturally described by a *closed* string theory. In this letter we propose to take this description seriously. We will present evidence that it indeed gives rise to a complete, U-duality invariant counting of BPS states. In our presentation we will focus on the central ideas of the construction, since the details of the calculations will be published elsewhere [13].

STRINGS ON THE FIVE-BRANE

For definiteness and simplicity, we will concentrate on *toroidal* compactifications of M-theory, because this will allow us to make maximal use of the space-time supersymmetries. Also, we consider only flat five-branes with the topology of T^5 . The five-brane represents a soliton configuration that breaks half of the 32 space-time supersymmetries of M-theory. On the $5 + 1$ dimensional world-volume the unbroken supercharges combine into 4 four-component chiral spinors that generate a $N = (4, 0)$ supersymmetry. It is known [14] that the *effective* world-brane theory is, after appropriate gauge-fixing, described by a tensor multiplet containing 5 scalars, an anti-symmetric tensor b^+ with self-dual three-form field strength db^+ , and 4 chiral fermions ψ . These fields must represent the massless states of the closed strings that live on the world-brane. One may think about these strings as solitons of the effective world brane theory that are charged with respect to the tensor b^+ . The self-duality of db^+ implies that we are dealing with a 6-dimensional *self-dual* string theory (cf. [15]).

The string configurations break half of the world-brane supersymmetries, and hence the world sheet formulation of this string must contain 8 supercharges: 4 left-moving and 4 right-moving. The 8 remaining supercharges, those that are broken by the string, give rise to 4 left-moving and 4 right-moving fermionic Goldstone modes λ and $\bar{\lambda}$. We assume that the world-sheet theory can be formulated in a light-cone gauge, and so one expects to have 4 bosonic fields x that describe the transversal directions inside the $5 + 1$ dimensional world-volume of the five-brane. We find it convenient to label these fields using chiral spinor indices a and \dot{a} of the transversal $SO(4)$ rotation group. In this notation the left-moving fields on the string world-sheet are

$$x^{a\dot{a}}(z), \lambda_{\alpha}^a(z), \quad (2)$$

and the right-moving fields are

$$x^{a\dot{a}}(\bar{z}), \bar{\lambda}_{\dot{\alpha}}^a(\bar{z}), \quad (3)$$

where we used the fact that the fermions λ and $\bar{\lambda}$ have the same chirality with respect to the $SO(4)$. The indices α ($\dot{\alpha}$) are (anti-)chiral spinor indices of another $SO(4)$ which, as will become clear, becomes identified with part of the space-time rotations. On the five-brane it will be realized as an R-symmetry.

We propose to take the fields $(x^{a\dot{a}}, \lambda_{\alpha}^a, \bar{\lambda}_{\dot{\alpha}}^a)$ as the complete field content of the world-sheet theory in the light-cone gauge, without worrying about a possible covariant formulation. In fact, we have essentially half of the world-sheet fields of the type IIB string in the Green-Schwarz formulation* and we can use this analogy to check that we get indeed the right massless fields. The ground states must form a multiplet of the left-moving

*A covariant formulation of this string is likely to have a gravitational anomaly, but this may be

zero-mode algebra $\{\lambda_a^\alpha, \lambda_b^\beta\} = \epsilon_{ab}\epsilon^{\alpha\beta}$. This gives 2 left-moving bosonic ground states $|\alpha\rangle$ and 2 fermionic states $|a\rangle$. By taking the tensor product with the right-moving vacua one obtains in total 16 ground states

$$\left(|\alpha, k_L\rangle \oplus |a, k_L\rangle\right) \otimes \left(|\beta, k_R\rangle \oplus |b, k_R\rangle\right). \quad (4)$$

Here we also took into account the momenta (k_L, k_R) , which form a $\Gamma_{5,5}$ lattice, since we have assumed that the five-brane has the topology of T^5 . Level matching implies that for the ground states one should have that $|k_L| = |k_R|$. Notice that these ground states are stable as long as the pair (k_L, k_R) is a primitive vector on $\Gamma_{5,5}$.

Ignoring for a moment the string winding numbers, we find that the ground states indeed represent the Fourier modes of the massless tensor multiplet on the five-brane. Specifically, the states $|\alpha\dot{\beta}\rangle$ describe four scalars $X^{\alpha\dot{\beta}} = \sigma_i^{\alpha\dot{\beta}} X^i$ that transform as a vector of the $SO(4)$ R-symmetry, while $|\alpha b\rangle$ and $|a\dot{\beta}\rangle$ describe the eight helicity states of the 4 world-brane fermions ψ^α and $\psi^{\dot{\alpha}}$. Finally, the RR-like states $|ab\rangle$ decompose into a fifth scalar, which we call Y , and the 3 helicity states of the tensor field b^+ . These are indeed the fields that parametrize the collective excitations of the five-brane soliton. In our description, however, these are just the low energy modes. The complete set of fluctuations of the five-brane are parametrized by the quantum string states. We also note that from the point of view of the string world-sheet the 5 scalars on the five-brane that describe the transverse oscillations naturally split up into four X^i 's plus one ‘‘RR’’-scalar Y . The interpretation of these fields in terms of the five-brane geometry will be discussed in detail in [13]. From this point of view it can be shown that Y is naturally compactified.

Just as in the type II superstring one can combine the 4 left-moving and 4 right-moving supercharges $G^{\dot{a}\alpha} = \oint \partial x^{\dot{a}b} \lambda_b^\alpha$ and $\bar{G}^{\dot{a}\dot{\alpha}} = \oint \bar{\partial} x^{\dot{a}b} \bar{\lambda}_b^{\dot{\alpha}}$ together with the eight fermion zero-modes of $\lambda^{a\alpha}$ and $\bar{\lambda}^{a\dot{\beta}}$ to construct the $N = (4, 0)$ supercharges \mathcal{Q} on the world-volume of the five-brane. The fact that the string lives on a 5-torus, however, has important consequences for the supersymmetry algebra. Namely, the anti-commutator of the left-moving supercharges produces the left-moving momentum k_L , while the right-movers give k_R . Hence the string states form representations of the $N = (4, 0)$ supersymmetry algebra[†]

$$\begin{aligned} \{\mathcal{Q}^{a\alpha}, \mathcal{Q}^{b\beta}\} &= \epsilon^{\alpha\beta}(\mathcal{P}^0 \mathbf{1}^{ab} + \mathcal{P}_L^m \Gamma_m^{ab}), \\ \{\mathcal{Q}^{a\dot{\alpha}}, \mathcal{Q}^{b\dot{\beta}}\} &= \epsilon^{\dot{\alpha}\dot{\beta}}(\mathcal{P}^0 \mathbf{1}^{ab} + \mathcal{P}_R^m \Gamma_m^{ab}), \end{aligned} \quad (5)$$

canceled by adding a term of the form $\int b^+ \text{tr} R^2$ to the world volume action of the five brane and by accompanying the world-sheet reparametrizations by a gauge transformation of the b^+ field. A similar mechanism is described in the last reference of [10].

[†]We use a Hamiltonian notation with only the spatial rotation group $SO(5)$ manifest. Hence from now on the indices a and \dot{a} are combined into a four-valued $SO(5)$ spinor index a . The index m corresponds to a $SO(5)$ vector.

where Γ_m^{ab} are $SO(5)$ gamma matrices. The operators \mathcal{P}^0 , \mathcal{P}_L^m and \mathcal{P}_R^m act on multi-string states that form the Hilbert space of the five-brane. The five-brane Hamiltonian \mathcal{P}^0 measures the energy of the collection of strings, while $\frac{1}{2}(\mathcal{P}_L^m + \mathcal{P}_R^m)$ measure the total momentum. But we see that the algebra also contains a vector central charge $\mathcal{P}_L^m - \mathcal{P}_R^m$, which measures the sum of the string winding numbers around the 5 independent one-cycles on the T^5 of the world-brane.

COUNTING MULTIPLE BPS STRINGS

The self-dual string is an interacting theory. It has no weak coupling limit, since its coupling is fixed by the self-duality relation. In the following, however, we will assume that for the purpose of counting the number of BPS-states, it will be an allowed procedure to treat it as a theory of non-interacting strings. The BPS-restriction should indeed limit the possible interactions that can take place. Furthermore, we will find that our degeneracy formulas will be consistent with previous results obtained from D-brane technology [18, 19, 5], as well as with U-duality.

Our aim is to count BPS states of the space-time theory that respect either $1/4$ or $1/8$ of the supersymmetries. On the world-brane this translates to a condition that either 8 or 4 of the 16 supercharges annihilate the states. The strongest BPS condition is obtained by demanding that

$$\varepsilon_{\alpha\alpha} \mathcal{Q}^{a\alpha} |\text{BPS}\rangle = 0, \quad (6)$$

for four independent spinors $\varepsilon_{\alpha\alpha}$, and at the same time imposing four similar relations for the supercharges $\mathcal{Q}^{a\dot{\beta}}$. By dropping these latter four relations one gets a weaker BPS condition that, as we will see, allows many more states. In fact, we can easily treat both cases in parallel.

Without loss of generality we can assume that $|\text{BPS}\rangle$ is an eigenstate of \mathcal{P}_L^m and \mathcal{P}_R^m with eigenvalue P_L^m and P_R^m . Since the condition (6) holds for all the states $|\text{BPS}\rangle$ in the same multiplet, we can use the algebra (5) to deduce that

$$\varepsilon_{\alpha\alpha} (P^0 \mathbf{1}^{ab} + P_L^m \Gamma_m^{ab}) = 0. \quad (7)$$

The equation (7) only has solutions when $P^0 = \pm |P_L|$. To count how many states we have for a given value of P_L^m we have to consider the multiple string states that have a total left-moving momentum $P_L^m \sim \sum k_L^m$ and total energy $|P_L|$. Since the energy of a single string is bounded from below by $|k_L|$, we deduce that in fact all momenta k_L^m must be in the *same* direction, namely that of P_L^m . For generic $\Gamma_{5,5}$ lattice this implies that (k_L, k_R) must be a multiple ℓ of the *primitive* vector $(\hat{P}_L, \hat{P}_R) \in \Gamma_{5,5}$ in the direction of (P_L, P_R) . Thus we have

$$(P_L, P_R) = N_P (\hat{P}_L, \hat{P}_R), \quad (8)$$

where N_P is an integer which is defined by this equation. Thus we see that in a BPS state of the five-brane, all world-brane momenta and windings of the strings are in the same direction. In other words, the BPS restriction implies that the five-brane behaves effectively as a 1+1-dimensional string-like object, as will become more evident in the following.

Now let \mathcal{H}_ℓ denote the space of single string BPS states with momentum $k = \ell \hat{P}$ that are in the left moving ground state. Via the level matching condition, its dimension is given by (here $\hat{P}^2 = |\hat{P}_L|^2 - |\hat{P}_R|^2$)

$$\dim \mathcal{H}_\ell = d\left(\frac{1}{2}\ell^2 \hat{P}^2\right), \quad (9)$$

which is the coefficient of the term $q^{\frac{1}{2}\ell^2 \hat{P}^2}$ in the elliptic genus of T^4

$$\sum_N d(N) q^N = 16 \prod_n \left(\frac{1+q^n}{1-q^n} \right)^4. \quad (10)$$

We will now make the assumption that we can treat the second quantized theory on the five-brane as a theory of non-interacting strings. The space of all multiple string states that satisfy the BPS conditions, and with a given total energy-momentum P , then takes the form

$$\mathcal{H}_P = \bigoplus_{\sum \ell N_\ell = N_P} \bigotimes_{\ell} \text{Sym}^{N_\ell} \mathcal{H}_\ell, \quad (11)$$

where Sym^N indicates that N -th symmetric tensor product. At first one may think that only the states with N_P “primitive” strings with $\ell = 1$ should contribute, because only these are stable. This would imply that only the first term $\text{Sym}^{N_P} \mathcal{H}_1$ must be kept. However, we propose that the correct interpretation of the Hilbert spaces \mathcal{H}_ℓ with $\ell > 1$ is that they represent the contributions of the various fixed points of the permutation group, and hence describe the bound states of ℓ different primitive strings. These therefore also represent stable states, and should be counted as well. We thus conjecture that the exact dimension of \mathcal{H}_P is determined by the character expansion[‡]

$$\sum_{N_P} q^{N_P} \dim \mathcal{H}_P = (16)^2 \prod_{\ell} \left(\frac{1+q^\ell}{1-q^\ell} \right)^{\frac{1}{2}d\left(\frac{1}{2}\ell^2 \hat{P}^2\right)}. \quad (12)$$

This formula somewhat resembles the expressions of Borchers [16] for the denominator formula of a generalized Kac-Moody algebra. Presumably, by a similar calculation as in

[‡]Here we included a prefactor $(16)^2$, which represents the dimension of the zero-mode Hilbert space on the five-brane world-volume.

[17] one can relate its logarithm to the one-loop amplitude of the self-dual string. We further note that for $\hat{P}^2 = 0$ (12) reduces to the standard (chiral) superstring partition function (since $\frac{1}{2}d(0) = 8$). This correspondence will be explained below.

An alternative but presumably equivalent description of the second quantized BPS string states on the five-brane is obtained by considering the sigma model on the “target space” $\sum_N \text{Sym}^N T^4$. The intuitive picture behind this representation is that the light-cone description of an N -string state may also be thought of as that of a single string state on the N -fold symmetric product of its target space.[§] This description should be further supported by the correspondence between the above formula for the dimension of \mathcal{H}_P and the term at order $q^{\frac{1}{2}N_P\hat{P}^2}$ in the expansion of the elliptic genus of the orbifold $\text{Sym}^{N_P} T^4$. In this way of looking at it one finds that the asymptotic growth equals that of states at level $h = \frac{1}{2}N_P\hat{P}^2$ in a unitary conformal field theory with central charge $c = 6N_P$. The standard degeneracy formula gives

$$\dim \mathcal{H}_P \sim \exp\left(2\pi\sqrt{\frac{1}{2}(P_L^2 - P_R^2)}\right). \quad (13)$$

This same result can be obtained directly from (12). We will now turn to the discussion of the space-time meaning of the momentum vector P .

U-DUALITY INVARIANT BPS SPECTRUM IN $D = 6$.

Let us now focus on the space-time interpretation of these BPS states in toroidal compactifications of M-theory. We consider situations in which at least 5 of the coordinates are compact so that the five-brane can wrap completely around an internal T^5 . We will also assume that at least 4 coordinates are uncompactified because these will be identified with the fields X^i on the five-brane. This leaves us with two cases: $d = 6$ and $d = 5$.

We start with $d = 6$. The space-time effective action for M theory compactified on T^5 is given by $N = (4, 4)$ six-dimensional supergravity that contains as bosonic fields, besides the metric, 5 anti-symmetric tensors, 16 gauge fields and 25 scalars. The scalars parametrize a $SO(5, 5)/SO(5) \times SO(5)$ manifold and hence the expected U-duality group is $SO(5, 5, \mathbf{Z})$. The $N = (4, 4)$ supersymmetry algebra is

$$\begin{aligned} \{Q_\alpha^a, Q_\beta^b\} &= \omega^{ab} \not{p}_{\alpha\beta}, \\ \{Q_\alpha^a, \bar{Q}_\beta^b\} &= \delta_{\alpha\beta} Z^{ab}, \end{aligned} \quad (14)$$

where $a, b = 1, \dots, 4$ are now $SO(5) \cong \text{Usp}(4)$ spinor indices and ω^{ab} is an anti-symmetric matrix, that will be used to raise and lower indices. The algebra contains 16 central

[§]This proposed representation of the multi-string Hilbert states was motivated by the construction described in [18, 5], where the same sigma model was used to encode all possible D-brane configurations.

charges that are combined in the 4×4 matrix Z^{ab} , where $a, b = 1, \dots, 4$ are again $SO(5)$ spinor indices. The central charge Z^{ab} forms a 16 component spinor of the $SO(5, 5, \mathbf{Z})$ U-duality group, and takes its values on an integral lattice whose shape is determined by the expectation values of the scalars.

Now let us look at the BPS states in the six-dimensional space-time theory that respect $1/4$ of the space-time supersymmetries. The mass of these BPS state can be determined from Z^{ab} by solving the eigenvalue equations

$$\begin{aligned}(Z^\dagger Z)^a{}_b \varepsilon_L^b &= m_{\text{BPS}}^2 \varepsilon_L^a, \\ (ZZ^\dagger)^a{}_b \varepsilon_R^b &= m_{\text{BPS}}^2 \varepsilon_R^a.\end{aligned}\tag{15}$$

Each of these equations has two independent solutions, as can be seen for example from the fact that the matrices $Z^\dagger Z$ and ZZ^\dagger must be of the form

$$\begin{aligned}(Z^\dagger Z)^{ab} &= (m_{\text{BPS}}^2 - 2|K_L|)\mathbf{1}^{ab} + 2K_L^m \Gamma_m^{ab}, \\ (ZZ^\dagger)^{ab} &= (m_{\text{BPS}}^2 - 2|K_R|)\mathbf{1}^{ab} + 2K_R^m \Gamma_m^{ab}.\end{aligned}\tag{16}$$

Since $|K_L| = |K_R|$ the combination (K_L, K_R) forms a null vector of $SO(5, 5, \mathbf{Z})$. As we will show below, BPS states with fixed (K_L, K_R) correspond to those multi-string states on the five-brane for which the total momentum and winding number vector (P_L, P_R) is equal to minus (K_L, K_R) . Since this implies that $P_L^2 - P_R^2 = 0$, the corresponding BPS states are necessarily made up from the string ground states with $|k_L| = |k_R|$.

To derive the relations (16) we must understand how the space-time supersymmetry algebra, including its central charge, is realized on the world-volume of the five-brane. We assume that the world-brane theory is formulated in a light-cone gauge, so that the $SO(5, 1)$ space-time Lorentz group is broken to the $SO(4)$ subgroup of transverse rotations. On the world-brane this group becomes identified with the R-symmetry, and so the previously introduced spinor indices $\alpha, \dot{\alpha}$ indeed correspond to the two spin representations of the space-time rotations. Notice that the space-time chirality is thus linked with the chirality on the string world-sheet. To construct the $N = (4, 4)$ space-time supersymmetry algebra on the world-brane we need to use the zero-mode algebra of the various world-brane fields. The transversal momentum $p^{\alpha\dot{\beta}}$ is as usual identified with the zero-modes of the canonical conjugate of the fields $X^{\alpha\dot{\beta}}$. Similarly, there are fermion zero-modes S_b^α and $S_a^{\dot{\beta}}$ that represent the broken part of the space-time supercharges. They can be normalized such that they satisfy the algebra

$$\{S_a^\alpha, S_b^{\dot{\beta}}\} = \epsilon^{\alpha\dot{\beta}} \omega_{ab}.\tag{17}$$

More surprisingly, all 16 central charges Z_{ab} appear also as zero modes, namely as the 10 fluxes of the self-dual three-form db^+ through the 3-cycles of T^5 , the 5 winding numbers

of the scalar Y , and the 5-flux of its dual $*dY$. Thus these charges are in one-to-one correspondence with the odd homology of T^5 , which naturally forms a spinor representation of $SO(5, 5, \mathbf{Z})$. To complete the space-time supersymmetry algebra, one also has to use the world-brane supercharges $\mathcal{Q}^{a\alpha}$, which act on the zero-modes as

$$\begin{aligned}\{\mathcal{Q}_a^\alpha, S_b^\beta\} &= \epsilon^{\alpha\beta} Z_{ab}, \\ \{\mathcal{Q}_a^\alpha, S_b^{\dot{\beta}}\} &= \omega_{ab} \not{p}^{\alpha\dot{\beta}}.\end{aligned}\tag{18}$$

From these equations it follows that \mathcal{Q}_a^α contains a zero-mode contribution $Z^{ab} S_{b\alpha} + \not{p}^{\alpha\dot{\beta}} S_{a\dot{\beta}}$ in addition to the non-zero-modes that we have been considering up to now. Then, from the world-brane supersymmetry algebra (5) one deduces that $\mathcal{P}^0 1^{ab} + \mathcal{P}_L^m \Gamma_m^{ab}$ also contains a zero mode contribution $\frac{1}{2}(p_i^2 1^{ab} + (Z^\dagger Z)^{ab})$.

We can now use this to re-analyse the BPS conditions on the five brane. In $d = 6$ we have the condition that on physical states the total space-like momentum on the five-brane vanishes, $\mathcal{P}_L^m = 0$.[¶] This implies that the oscillator contribution P_L^m and the zero-mode contribution $K_L^m = \frac{1}{8}\text{tr}(\Gamma^m Z^\dagger Z)$ cancel, so that we derive that $P_L^m = -K_L^m$. Similarly one finds that $P_R^m = -K_R^m$. The BPS mass formula now follows by imposing the five-brane mass-shell condition $\mathcal{P}^0 = p_+ p_-$. This gives $m_{\text{BPS}}^2 = \frac{1}{4}\text{tr}(Z^\dagger Z) + 2|K|$, where $|K| = |K_L| = |K_R|$. Thus the $SO(5, 5)$ vector (P_L, P_R) satisfies $P_L^2 - P_R^2 = 0$, and hence is indeed a null-vector.

The degeneracy formula follows now immediately by specializing the general formula (12) to the case $\hat{P}^2 = 0$. So the number of BPS states is given by $D(N_K)$, where N_K is the number of times that the primitive vector \hat{K} fits in K , and $D(N)$ is the degeneracy at level N of the standard superstring partition function

$$\sum_N D(N) q^N = (16)^2 \prod_n \left(\frac{1 + q^n}{1 - q^n} \right)^8.\tag{19}$$

This degeneracy formula is the unique U-duality invariant extension of the results obtained in [19, 18] from the counting of string and D-brane BPS states. Note that pure string or D-brane configurations carry at most 8 charges that transform as an $SO(4, 4)$ vector (q_L, q_R) , and their degeneracy is $D(\frac{1}{2}(q_L^2 - q_R^2))$. The above result shows that for more general bound states between string and D-branes, the degeneracy is obtained by generalizing the T-duality invariant $\frac{1}{2}(q_L^2 - q_R^2)$ to the greatest common divisor of the ten components of the $SO(5, 5)$ vector (K_L, K_R) defined in (16).

[¶]This can be seen as a generalisation of the $L_0 - \bar{L}_0 = 0$ condition in light-cone string theory.

BPS SPECTRUM AND BLACK HOLE ENTROPY IN $D = 5$.

It is straightforward to extend the six-dimensional results to $d = 5$ and the BPS states that respect only $1/8$ of the space-time supersymmetries.

The 5 anti-symmetric tensor fields in 6 dimensions can be decomposed into 5 fields B_L^m with self dual field strength and 5 fields B_R^m with anti-self dual field strength. Together (B_L^m, B_R^m) form a vector of $SO(5, 5, \mathbf{Z})$. After further compactification to $d = 5$ these anti-symmetric tensor fields produce 10 additional gauge fields, with 10 corresponding charges (W_L^m, W_R^m) . Together with the Kaluza-Klein momentum p and the 16 charges contained in Z^{ab} this gives a total of 27 charges. These charges combine into one irreducible representation of the $E_{6(6)}(\mathbf{Z})$ U-duality group: an 8×8 pseudo-real, anti-symmetric and traceless matrix \mathcal{Z} [20]. This matrix can be expressed in terms of the $SO(5, 5)$ vector (W_L, W_R) , scalar p and spinor Z as

$$\mathcal{Z} = \begin{pmatrix} p + W_L \cdot \Gamma & Z \\ Z^\dagger & -p + W_R \cdot \Gamma \end{pmatrix}. \quad (20)$$

By dimensional arguments, the extremal black holes that carry these charges are expected to have a non-zero entropy that is the square root of a cubic expression in \mathcal{Z} . There is one unique cubic $E_{6(6)}$ invariant, namely $\text{tr} \mathcal{Z}^3$. In the above normalisation, this leads to the following prediction for the U-duality invariant entropy formula

$$\begin{aligned} S(\mathcal{Z}) &= 2\pi \left[\frac{1}{2}p(W_L^2 - W_R^2) + \frac{1}{8}W_L \cdot \text{tr}(\Gamma Z^\dagger Z) - \frac{1}{8}W_R \cdot \text{tr}(\Gamma Z Z^\dagger) \right]^{1/2} \\ &= 2\pi \left[\frac{1}{2p}(|pW_L - K_L|^2 - |pW_R - K_R|^2) \right]^{1/2} \end{aligned} \quad (21)$$

with K as defined in (16).

How can such a result be derived from the five-brane? Because we are considering $1/8$ BPS states in space-time, in this case we will only have to impose the left-moving BPS conditions. So in terms of the self-dual string one expects to get multiple string states on the five-brane with right-moving oscillators. Note however, that this is only possible when the momentum (P_L^m, P_R^m) is no longer a null vector, but satisfies $P^2 > 0$. Also, we need to represent the additional charges (W_L^m, W_R^m) and the Kaluza-Klein momentum p on the Hilbert states of the five-brane. The interpretation of the vector charges (W_L^m, W_R^m) is roughly that they indicate the winding number of the five-brane around the extra compactification circle. In this way the Kaluza-Klein momentum p is fed into the five-brane and this modifies the level matching relations on the five-brane to $\mathcal{P}_L^m = pW_L^m$ and similarly for the right-movers.^{||} Now following the same steps as before one finds that the

^{||}The analogous statement in string theory is that $L_0 - \bar{L}_0$ no longer vanishes when the space-like direction of the light-cone (x^+, x^-) plane is taken to be compact. Instead it equals pw with p and w the momentum and winding number in that direction.

total left-moving momentum of the multiple string states is

$$P_L^m = pW_L^m - K_L^m \quad (22)$$

and similarly for P_R^m .

Even though we are able to represent all 27 charges, only the $SO(5, 5, \mathbf{Z})$ subgroup of the U-duality group $E_{6(6)}(\mathbf{Z})$ is a manifest symmetry of the five-brane. The full U-duality is hidden in the light-cone construction, which has extra subtleties because we work on a “light-cone cylinder.” Because of this, our construction is in fact most straightforward for $p = 1$. In this case the counting of the states as explained above equation (13) goes through without change, and leads for the statistical entropy $S = \log(\dim \mathcal{H}_P)$ to the following result

$$S = 2\pi\sqrt{\frac{1}{2}(W - K)^2}. \quad (23)$$

Comparing to the expected $E_{6(6)}$ -invariant result (21) we see that for $p \neq 1$ an extra factor $1/p$ enters. In the light-cone gauge constructions that we have been using, this factor naturally follows from the relative normalisation of the time-coordinates on the world-sheet of the self-dual string and space-time. In this way we find that the five-brane description of the BPS states exhibits a maximal symmetry under the five-dimensional U-duality group. The fact that we furthermore reproduce the expected black hole entropy is a strong indication that the five-brane also gives an exhaustive description of the BPS spectrum in $d = 5$. It would be interesting to find a general covariant derivation of this result, starting from a formulation as in [21], since this could make the full $E_{6(6)}$ -invariance manifest from the start.

CONCLUDING REMARKS

We have shown that at least down to 5 dimensions the string formulation of the five-brane theory gives a unified description of all BPS states that is invariant under the U-duality group. The general quantisation of five-dimensional extended objects is still an unsolved problem, but we see convincing evidence that the “BPS quantisation” of the five-brane reduces it effectively to a type II string, since the momenta k^m on the world-brane are aligned with the central charge K^m . U-duality permutes these various strings, similar to the action of $SL(2, \mathbf{Z})$ on the (p, q) strings in type IIB string theory in ten dimensions. The correspondence with the type II string also shows immediately that our description for the spectrum of 1/4 BPS states is fully Lorentz invariant. This is however somewhat less obvious for the 1/8 BPS states.

We can also consider compactifications of M-theory on other manifolds than tori, such as orbifolds [10]. Particular cases that we have analyzed (see [13]) are $K3 \times S^1$ and

$T^4 \times (S^1/\mathbf{Z}_2)$. In both of these compactifications the five-brane can be shown to behave as a heterotic string. In the latter case, the geometric explanation is that the worldbrane geometry of the five-brane must also be of the form of $K3 \times S^1$. This result can be used to give a new explanation from M-theory of the various heterotic string/string dualities [22]. Furthermore, the counting of BPS states can be done in similar manner as in this letter [13], and is in agreement with the black hole entropy formula derived in [5].

Finally, let us comment on the generalization to four dimensions. Here one gets one extra electric charge p' from the Kaluza-Klein momentum, which gives a total of 28 charges. But a general state in $d = 4$ can carry 28 magnetic and 28 electric charges that combine into the 56 of the $E_{7(7)}(\mathbf{Z})$ U-duality group. In this case the entropy is expected to scale with the square root of a quartic expression in the central charge, which is now a general complex anti-symmetric 8×8 matrix. The unique quartic invariant is the so-called diamond function \diamond . In [23] this was used to conjecture that the macroscopic entropy in four dimensions equals $S = 2\pi\sqrt{\diamond}$. For the states obtained from the five-brane, this entropy formula can be expressed in terms of $SO(5, 5, \mathbf{Z})$ representations and takes the form

$$S = 2\pi\sqrt{\frac{p'}{2p}(pW - K)^2}. \quad (24)$$

We believe that via a straightforward generalization of the above procedure one should be able to reproduce this result. This would give a complementary derivation of the recently obtained results [7]. It seems, however, that to obtain a fully $E_{7(7)}$ -invariant description from the five-brane, new ingredients may be needed. Possibilities are the inclusion of D-brane configurations on the world-volume [24], or of bound states of several five-branes [11, 25], which would lead to non-abelian extensions of the self-dual string theory considered here.

ACKNOWLEDGEMENTS

We would like to thank S. Ferrara, C. Kounnas, W. Lerche, R. Minasian, and C. Vafa for interesting discussions and helpful comments. This research is partly supported by a Pionier Fellowship of NWO, a Fellowship of the Royal Dutch Academy of Sciences (K.N.A.W.), the Packard Foundation and the A.P. Sloan Foundation.

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